

# The Lanczos potential for Weyl-candidate tensors exists only in four dimensions

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## Abstract

We prove that a Lanczos potential  $L_{abc}$  for the Weyl candidate tensor  $W_{abcd}$  does not generally exist for dimensions higher than four. The technique is simply to assume the existence of such a potential in dimension  $n$ , and then check the integrability conditions for the assumed system of differential equations; if the integrability conditions yield another non-trivial differential system for  $L_{abc}$  and  $W_{abcd}$ , then this system's integrability conditions should be checked; and so on. When we find a non-trivial condition *involving only  $W_{abcd}$  and its derivatives*, then clearly Weyl candidate tensors failing to satisfy that condition cannot be written in terms of a Lanczos potential  $L_{abc}$ .

## 1 Introduction

*The mere presence of an unadulterated Riemannian geometry of specifically 4-dimensions brings into existence a tensor of third order of 16 components, which bridges the gap between the second-order tensor of the line element and the fourth order tensor of the Riemannian curvature.* C. Lanczos (1962) Rev. Mod. Phys. **34**, 379.

Even though there is a deeper understanding today of the 3-tensor potential for the Weyl tensor proposed by Lanczos [7], there exists still a question mark over whether its existence is unique to 4-dimensions, as seems to be claimed by Lanczos.

In this paper we demonstrate that it is not possible, in general, to obtain a Lanczos potential for all Weyl candidates in spaces of dimension  $n$ , where  $n > 4$ ; this does not prove unambiguously the truth of Lanczos' claim [for the *Weyl curvature tensor*], but certainly emphasises that four dimensional spaces play a very special role regarding the existence of Lanczos potentials.

Lanczos proposed that, in four dimensions, the Weyl tensor  $C_{abcd}$  could be given locally in terms of a potential  $L_{abc}$  as

$$C^{ab}_{cd} = L^{ab}_{[c;d]} + L_{cd}^{[a;b]} - {}^*L^{*ab}_{[c;d]} - {}^*L_{cd}^{*[a;b]} \quad (1)$$

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where

$$L_{abc} = L_{[ab]c} \quad (2a)$$

$$L_{[abc]} = 0 \quad (2b)$$

and  $*$  denotes the usual Hodge dual. Bampi and Caviglia [2] have shown that, although his proof was flawed, Lanczos' result is still valid. Moreover, Bampi and Caviglia [2] have shown that a Weyl candidate, i.e. *any* 4-tensor  $W_{abcd}$  having the index symmetries

$$W_{abcd} = W_{[ab]cd} = W_{ab[cd]}, \quad (3a)$$

$$W_{abcd} = W_{cdab} \quad (3b)$$

$$W_{a[bcd]} = 0 \quad (3c)$$

$$W^a{}_{bad} = 0 \quad (3d)$$

can be given locally in such a form, in four dimensions. Illge [6] has given equivalent existence results (as part of a larger investigation), using spinors, in 4-dimensional spacetimes with Lorentz signature. Andersson and Edgar [1] has also given an existence proof in spinors. They have also translated that proof into tensors in four dimensions; to modify this proof to other dimensions seems impossible since some identities valid only in four dimensions were needed in that proof.

Although the form given in (1) clearly cannot be generalised directly to other dimensions, yet by writing (1) in an equivalent form, without Hodge duals, it is straightforward to generalise this equivalent form of (1) to  $n > 4$  dimensions,

$$\begin{aligned} W^{ab}{}_{cd} = & 2L^{ab}{}_{[c;d]} + 2L_{cd}^{[a;b]} \\ & - \frac{4}{n-2} \delta_{[c}^{[a} \left( L^{b]i}{}_{d];i} - L^{b]i}{}_{[i];d]} + L_{d]i}{}^{b];i} - L_{d]i}{}^{[i];b]} \right) \\ & + \frac{8}{(n-2)(n-1)} \delta_{[c}^a \delta_{d]}^b L^{ij}{}_{i;j} \end{aligned} \quad (4)$$

where  $W_{abcd}$  and  $L_{abc}$  satisfy (3) and (2) respectively, (Bampi and Caviglia [2]).

This can be further simplified to

$$W^{ab}{}_{cd} = 2L^{ab}{}_{[c;d]} + 2L_{cd}^{[a;b]} - \frac{4}{(n-2)} \delta_{[c}^{[a} \left( L^{b]i}{}_{d];i} + L_{d]i}{}^{b];i} \right) \quad (5)$$

by the gauge choice

$$L_{ab}{}^b = 0 \quad (6)$$

which is called the Lanczos algebraic gauge. To see that this really is a gauge choice, rather than a restriction, we can take any  $W_{abcd}$  and any  $L_{abc}$  that satisfies (4), then  $L'_{abc} = L_{abc} + \Phi_{[a} g_{b]c}$  also satisfy (4) for all  $\Phi_a$ . This is

easily shown by inserting  $L'_{abc}$  into (4). The particular choice  $\Phi_a = -\frac{2}{n-1}L_{ab}{}^b$  gives the Lanczos algebraic gauge. In the rest of this paper this gauge choice as assumed.

From the existence proofs in four dimensions it follows that there is a further gauge choice

$$L_{abc}{}^{;c} = \xi_{ab} \quad (7)$$

where  $\xi_{ab}$  is any antisymmetric tensor. When  $\xi_{ab} = 0$  this choice is called the Lanczos differential gauge. Whether this gauge freedom remains in higher dimensions or not is not known. Therefore we have not assumed anything about this gauge in this paper.

Lanczos gave no direct support for his claim regarding the privileged role of four dimensions, although there is some indirect evidence (see Comment 7 in section 4) that Lanczos potentials may not exist in higher dimensions. So we now consider the question whether *every* 4-tensor  $W_{abcd}$  having the index symmetries (3) can be given locally in the form (4) in terms of some  $L_{abc}$  with the properties (2), *for all spaces of dimension  $n$ , where  $n > 4$ .*

The case when  $n < 4$  is not considered. Any tensor  $W_{abcd}$  with the symmetries (3a) and (3d) is identically zero in those cases. The right hand side of (4) also has those symmetries whenever  $L_{abc}$  has the symmetry (2a). Therefore, any tensor  $L_{abc}$  which is antisymmetric over its first two indices satisfies (4) identically when  $n < 4$ .

## 2 Lanczos potentials for Weyl candidates in Flat Space

We start our considerations in flat space. This investigation is important in its own right, but it also enables us to get a pattern for how to proceed in curved space.

### 2.1 Six dimensions and higher

By differentiating (5) twice we can obtain,

$$W^{[ab}{}_{[cd;e]}{}^{f]} = -\frac{4}{n-2}\delta_{[c}^{[a}\left(L^{b|z|}{}_{d;|z|e]}{}^{f]} + L_d{}^{[z|b}{}_{;|z|e]}{}^{f]}\right) \quad (8)$$

Contracting this once gives

$$\begin{aligned} W^{[ab}{}_{[cd;e]}{}^{i]} &= -\frac{4(n-4)}{9(n-2)}\left(L^{[a|i|}{}_{[c;|z|d]}{}^{b]} + L_{[c}{}^{i[a}{}_{;|z|d]}{}^{b]}\right) \\ &\quad - \frac{4}{9(n-2)}\delta_{[c}^{[a}\left(L^{b]i}{}_{d;ij}{}^j + L_d{}^{[i|b]}{}_{;ij}{}^j\right. \\ &\quad \left.- L^{[ji]}{}_{d;ij}{}^{b]} - L_d{}^{[ji]}{}_{;ij}{}^{b]} - L^{b]i}{}_{[j;|z|d]}{}^j - L_{[j}{}^{[i|b]}{}_{;i|d]}{}^j\right) \end{aligned} \quad (9)$$

and once more gives

$$W^{ai}{}_{cj;i}{}^j = \frac{(n-3)}{(n-2)} \left( L^{ai}{}_{c;ij}{}^j + L_c{}^{ia}{}_{;ij}{}^j - L^{ji}{}_{c;ij}{}^a - L_c{}^{ij}{}_{;ij}{}^a - L^{ai}{}_{j;ic}{}^j - L_j{}^{ia}{}_{;ic}{}^j \right) \quad (10)$$

This can be substituted into (9) to give

$$W^{[ab}{}_{[cd;i]}{}^i] = -\frac{4(n-4)}{9(n-2)} \left( L^{[a|i}{}_{[c;|i|d]}{}^{b]} + L_{[c}{}^{i[a}{}_{;|i|d]}{}^{b]} \right) - \frac{4}{9(n-3)} \delta_{[c}^{[a} W^{b]i}{}_{d]j;i}{}^j \quad (11)$$

which in turn can be substituted into (8) to give

$$W^{[ab}{}_{[cd;e]}{}^f] = \frac{1}{n-4} \delta_{[c}^{[a} W^{b]f}{}_{de];i}{}^i + \frac{2}{n-4} \delta_{[c}^{[a} W^{i|b}{}_{de];i}{}^f] + \frac{2}{n-4} \delta_{[c}^{[a} W^{b]f}{}_{|i|d;e]}{}^i + \frac{4}{(n-3)(n-4)} \delta_{[c}^{[a} \delta_d^{b]} W^{f]i}{}_{e]j;i}{}^j \quad (12)$$

This equation can be rewritten in the form  $T^{[abf]}{}_{[cde]} = 0$  and it is easy to show that the left hand side is trace free, i.e.  $T^{[abe]}{}_{[cde]} = 0$ . It has been shown by Lovelock [8] that for such trace-free tensors the equation

$$T^{[abf]}{}_{[cde]} = 0 \quad (13)$$

is trivially satisfied in five dimensions (and less), i.e. it is a simple algebraic identity. It can also be shown by working out the identity

$$0 = T^{ijk}{}_{[cde]} \delta_i^a \delta_j^b \delta_k^f \quad (14)$$

which is true in five dimensions (and less) because antisymmetrising over more than five indices makes the expression vanish; then the trace-free properties give (13). Equation (12) is therefore not a constraint in five dimensions.

To see that it is an effective restriction in six dimensions (and higher) we can choose a local Cartesian coordinate system and its associated basis. We then examine the component corresponding to  $(a, b, f, c, d, e) = (1, 2, 3, 4, 5, 6)$ . Then equation (12) becomes

$$W^{12}{}_{[45,6]}{}^3 = 0 \quad (15)$$

Choosing  $W^{12}{}_{45} = W^1{}_4 W^2{}_5 = \sin(x_3) \sin(x_6)$  and all other components zero except for those needed to give the right symmetries (3) for  $W_{abcd}$ , then the left hand side becomes  $\pm \cos(x_3) \cos(x_6)$  (the sign depends on the signature of the metric) which is clearly not zero.

## 2.2 Five dimensions (and higher)

If we now differentiate (11) once again and antisymmetrise on all free upper indices we obtain

$$W^{[ab}_{cd;i}{}^{i|e]} + 2W^{[ab}_{i[c;d]}{}^{i|e]} + \frac{4}{(n-3)}\delta^{[a}_{[c}W^{b|i}_{d]j;i}{}^{j|e]} = 0 \quad (16)$$

It is easy to show that the left hand side is *not* identically zero in  $n > 4$  dimensions. One straightforward way is simply to choose a local Cartesian coordinate system and its associated basis and choosing  $W^{12}_{34} = W^1_3{}^2_4 = \sin(x_5)$  and all other components zero except for those needed to give the right symmetries. Then the component on the left hand side corresponding to  $(a, b, c, d, e) = (1, 2, 3, 4, 5)$  becomes

$$W^{12}_{34;5}{}^{55} = -\cos(x_5) \quad (17)$$

which is not zero.

Another more interesting method is to note that for the subclass of tensors satisfying  $W^a_{bcd;a} = 0$ , the constraint (16) reduces to

$$W^{ab}_{[cd;i}{}^{i|e]} = 0 \quad (18)$$

which is clearly not identically zero, in  $n > 4$  dimensions, even when  $W^a_{bcd;a} = 0$ . So, by virtue of (4), there is an effective restriction (16) imposed on  $W_{abcd}$  in dimensions  $n > 4$ , in flat space.

## 2.3 Four dimensions

In four dimensions, there can be no restrictions on which tensors  $W_{abcd}$  possesses a potential since, as stated in section 1, in four dimensions any such tensor with the symmetries (3) can be given by a Lanczos potential. However, it is not immediately obvious from discussion above that the expressions presented here are not constraints in four dimensions, as we require.

The constraint (12) is not valid in four dimensions because of factors of  $n-4$  in the denominators. However, equation (11) does not contain any  $L_{abc}$  and is valid in four dimensions. On the other hand (11) can be rearranged as

$$\left(W^{[ab}_{[cd}\delta^{i]}_{j]}\right)_{;i}{}^j = 0. \quad (19)$$

It can be shown that the left hand side is also identically zero; this is because the expression in parenthesis  $\left(W^{[ab}_{[cd}\delta^{i]}_{j]}\right)$  belongs to a special class of dimensionally dependent identities found by Lovelock [8]; these are known to be satisfied *in four dimensions*.

Hence (11) is identically satisfied in four dimensions, and since (16) is simply a derivative of (11) it is also identically zero in four dimensions.

### 3 Lanczos potentials for Weyl candidates in curved space

#### 3.1 Six dimensions and higher

It is possible to carry through the same steps that lead to (12) but in curved space. Then we get (12) plus product terms of  $L_{abc}$  and curvature tensors. By rearranging we are able to use (5) to eliminate some  $L_{abc}$ , but unfortunately not all; there remain product terms involving  $L_{abc}$  and Weyl curvature tensors:

$$\begin{aligned}
W^{[ab]}_{[cd;e]f} &= L_{[cd}^{[a;i]C^{bf]}_{e]i} - L_{[cd}^{i;[aC^{bf]}_{e]i} - 2L_{[c}^{i[a;dC^{bf]}_{e]i} - 2L_{[c}^{i[a;bC^f]_{|i|de]} \\
&\quad - 2L^{[a|i]_{[c}{}^{bC^f]_{|i|de]} + L_{[c}^{i[aC^{bf]}_{de];i} + L^{[a|i]_{[c}C^{bf]}_{de];i} \\
&\quad - \frac{2}{n-4}\delta_{[c}^{[a}\left( L_{de]}^{i;j|}C^{bf]}_{ij} + 2L_d^{i|b;|j|}C^f_{|j|e]i} + 2L^{b|i|_d}{}^{j|}C^f_{|i|e]j} \right. \\
&\quad \quad + L_d^{i|j|}{}_{;e]}C^{bf]}_{ij} + L^{b|i;j|}{}_{;f]}C_{de]ij} + L_d^{i|j|}{}_{;j}C^{bf]}_{e]i} + L^{b|i;j|}{}_{;|j|}C^f_{|i|de]} \\
&\quad \quad + L_d^{i|j|}{}_{;i}C^{bf]}_{e]j} + L^{i|j|}{}_{;d}{}^{bC^f]}_{e]ij} - L^{i|j|b}{}_{;d}C^f_{e]ij} + L^{i|j|b}{}_{;|i|}C^f_{|j|de]} \\
&\quad \quad - L^{i|j|b;f]}C_{de]ij} + L_d^{i|j|}C^{bf]}_{e]j;i} + L^{b|i;j|}C_{de]}{}^f{}_{j;i} - \frac{1}{2}L^{i|j|}{}_dC^{bf]}_{|ij|e]} \\
&\quad \quad \left. - \frac{1}{2}L^{i|j|b}C_{de]ij};^f + \frac{2}{n-3}L_d^{i|b}C^f_{|i|e];j} + \frac{2}{n-3}L^{b|i|_d}C_{e]i}{}^f{}_{j;j} \right) \\
&\quad - \frac{2}{(n-3)(n-4)}\delta_{[c}^{[a}\delta_d^{b}\left( 2L_e^{i;j;k|}C^f_{jki} + 2L^{f]ij;k}C_{i[jk|e]} - L^{i|j|}{}_{;e]}C_{i[jk|e]}{}^f \right. \\
&\quad \quad - 2L^{i|j|f];k}C_{i[jk|e]} + 2L^{i|j;k|}{}_{;e]}C_{ijk}{}^f + 2L^{i|j;k|}{}_{;f]}C_{i[jk|e]} + L^{i|j;k|}C_{i[jk|e]}{}^f{}_{;j} \\
&\quad \quad + L^{i|j;k|}C_{i[jk|e]}{}^f{}_{;e]} + \frac{2}{n-3}L_e^{i|j|}C^f_{i[jk];k} + \frac{2}{n-3}L^{f]ij}C_{e]ijk};^k \\
&\quad \quad \left. + \frac{n-2}{n-3}L^{i|j|}{}_{;e]}C_{ij}{}^f{}_{k};^k + \frac{n-2}{n-3}L^{i|j|f]}C_{i[j|e]k};^k \right) \\
&\quad + \frac{4}{(n-3)^2(n-4)}\delta_{[c}^{[a}\delta_d^{b}\delta_e^{f]}L^{ijk}C_{ijkl};^l \\
&\quad + \frac{1}{(n-2)(n-3)(n-4)}\delta_{[c}^{[a}\delta_d^{b}\delta_e^{f]}C_{ijkl}W^{ijkl} \\
&\quad + \frac{1}{n-4}\delta_{[c}^{[a}W^{bf]}_{de];i}{}^i + \frac{2}{n-4}\delta_{[c}^{[a}W^{i|b}{}_{de];i}{}^f + \frac{2}{n-4}\delta_{[c}^{[a}W^{bf]}_{|i|d;e]}{}^i \\
&\quad + \frac{4}{(n-3)(n-4)}\delta_{[c}^{[a}\delta_d^{b}W^f{}^i{}_{e]j};^j - \frac{2}{n-2}W^{[a}{}^b{}_{[cd}\tilde{R}^f]_{e]} \\
&\quad - \frac{4}{(n-2)(n-4)}\delta_{[c}^{[a}W^{b|i|f]}{}_d\tilde{R}_{e]i} - \frac{4}{(n-2)(n-4)}\delta_{[c}^{[a}W^b{}_{de]}{}^{i|}\tilde{R}_f{}_{i} \\
&\quad - \frac{4}{(n-2)(n-3)(n-4)}\delta_{[c}^{[a}\delta_d^{b}W^f{}^i{}_{e]j}{}^j\tilde{R}_{ij}
\end{aligned} \tag{20}$$

where  $\tilde{R}_{ab}$  is the trace free Ricci curvature tensor and  $R$  the scalar curvature.

Restricting our considerations to conformally flat manifolds, i.e.  $C_{abcd} = 0$ , gives

$$\begin{aligned}
 W^{[ab}_{[cd;e]}{}^{f]} &= \frac{1}{n-4} \delta_{[c}^{[a} W^{bf]}_{de];i}{}^i + \frac{2}{n-4} \delta_{[c}^{[a} W^{b|i|}_{de];i}{}^{f]} \\
 &+ \frac{2}{n-4} \delta_{[c}^{[a} W^{bf]}_{|i|d;e]}{}^i + \frac{4}{(n-3)(n-4)} \delta_{[c}^{[a} \delta_d^b W^{f]i}_{e]j;i}{}^j \\
 &- \frac{2}{n-2} W^{[a}_{[cd}{}^b \tilde{R}^{f]}_{e]} - \frac{4}{(n-2)(n-4)} \delta_{[c}^{[a} W^{b|i|}_{d]} \tilde{R}_{e]i} \quad (21) \\
 &- \frac{4}{(n-2)(n-4)} \delta_{[c}^{[a} W^b_{de]}{}^{i|i} \tilde{R}^{f]}_{i} \\
 &- \frac{4}{(n-2)(n-3)(n-4)} \delta_{[c}^{[a} \delta_d^b W^{f]i}_{e]}{}^j \tilde{R}_{ij}.
 \end{aligned}$$

where we note that all the terms containing  $L_{abc}$  explicitly have disappeared.

As in the flat case, this expression can be rearranged in the form  $T^{[abf]}_{[cde]} = 0$  where  $T^{[abf]}_{[cde]}$  is trace free, so this is not a restriction in five dimensions. When specialised to flat space this restriction is the same as (12) so we know that this is an effective restriction in six dimensions and higher even though we do not know if this is an effective restriction for *all* conformally flat manifolds.

### 3.2 Five dimensions (and higher)

Carrying through the same steps that led to (16) but in a general curved space gives an expression that is unmanageable in size. However, if we assume constant curvature, i.e.  $C_{abcd} = 0$  and  $\tilde{R}_{ab} = 0$ , we do get a manageable restriction. With those assumptions we get

$$\begin{aligned}
 W^{[ab}_{cd;i}{}^{i|i|e]} + 2W^{[ab}_{i[c;d]}{}^{i|i|e]} + \frac{4}{n-3} \delta_{[c}^{[a} W^{b|i}_{d]j;i}{}^{j|e]} \\
 - \frac{2(n-4)}{n(n-1)} R W^{[a}_{c}{}^b{}_{d;e]} + \frac{4}{n(n-1)} R \delta_{[c}^{[a} W^{b|i|}_{d]}{}^{i|e]} = 0 \quad (22)
 \end{aligned}$$

where  $R$  is the scalar curvature and we note that all the terms containing  $L_{abc}$  explicitly have disappeared.

When specialised to flat space this restriction is the same as (16) and we can therefore conclude that this is an effective restriction in five dimensions and higher. However, we do not know whether this is an effective restriction for *all* constant curvature spaces.

The restrictions (21) and (22) can be somewhat rearranged if the symmetry (3c) is used but the effect is only cosmetic, see comment 6 in next section.

## 4 Summary and discussion

What has been proved here can be summarised in the following theorems.

**Theorem 1** *Let  $M$  be a metric  $n$ -dimensional differentiable flat manifold,  $n \geq 6$ , with any signature. Then tensors  $W_{abcd}$  — with the properties (3) — which fail to satisfy (12) cannot be given via (4) by a three times differentiable potential  $L_{abc}$  with the properties (2).*

**Theorem 2** *Let  $M$  be a metric  $n$ -dimensional differentiable flat manifold,  $n \geq 5$ , with any signature. Then tensors  $W_{abcd}$  — with the properties (3) — which fail to satisfy (16) cannot be given via (4) by a four times differentiable potential  $L_{abc}$  with the properties (2).*

**Theorem 3** *Let  $M$  be a metric  $n$ -dimensional differentiable manifold,  $n \geq 6$ , with zero Weyl curvature and any signature. Then tensors  $W_{abcd}$  — with the properties (3) — which fail to satisfy (21) cannot be given via (4) by a three times differentiable potential  $L_{abc}$  with the properties (2).*

**Theorem 4** *Let  $M$  be a metric  $n$ -dimensional differentiable manifold,  $n \geq 5$ , with constant curvature and any signature. Then tensors  $W_{abcd}$  — with the properties (3) — which fail to satisfy (22) cannot be given via (4) by a four times differentiable potential  $L_{abc}$  with the properties (2).*

In the above calculations the properties (2b) and (3c) have not been used. Therefore we can also conclude that in these theorems we can relax the condition on  $W_{abcd}$  to (3a,b,d) and on  $L_{abc}$  to (2a). This more general result is of importance when we compare our results with the results of Bampi and Caviglia [2] for the parallel problem discussed in Comment 6 below.

A number of points should be emphasised:

1. One must, of course, pose the question whether there are other possible generalisations of (1) which would be more successful than (4). Making the assumptions that the potential should be a three index tensor and having the symmetries (2), then already the very simple and most obvious choice

$$W_{abcd} = 4L_{abc;d} \tag{23}$$

leads to the same generalisation via the following steps. The antisymmetry of the last two indices gives

$$W_{abcd} = W_{ab[cd]} = 4L_{ab[c;d]} \tag{24}$$



The symmetry (3b) gives

$$W_{abcd} = \frac{1}{2}W_{abcd} + \frac{1}{2}W_{cdab} = 2L_{ab[c;d]} + 2L_{cd[a;b]} \quad (25)$$

Subtracting off the trace of this equation gives us (4).

It is difficult to imagine another generalisation which cannot be reduced to (4).

**2.** The most obvious way to see that there exist some  $W_{abcd}$  which have a potential is to choose a tensor  $L_{abc}$  and then *define* a tensor  $W_{abcd}$  via (4). Obviously the  $W_{abcd}$  so obtained has this Lanczos potential  $L_{abc}$ .

Of course this result does not rule out the possibility of *some* significant subclasses of these tensors  $W_{abcd}$  having such a potential — even in flat spaces. For instance, suppose that a tensor  $B_{abcd}$ , with the properties (3a) and (3c), also satisfies the Bianchi-like equation  $B_{ab[cd;e]} = 0$ ; in flat space, in all dimensions, it is well known that such tensors  $B_{abcd}$  *with this additional property* can be written as  $B_{abcd} = 2h_{a[c;d]b} - 2h_{b[c;d]a}$ . Consider next the case where  $B_{abcd}$  is a Weyl candidate i.e.,  $B_{abcd} \equiv W_{abcd}$ , and it follows that  $W_{abcd}$ , satisfying (3) and the additional property  $W_{ab[cd;e]} = 0$ , can always be written in the form (4) with the appropriate 3-potential  $L_{abc} = h_{c[a;b]}$ . Clearly for  $W_{abcd}$  with such properties, the constraints (12) and (16) is trivially satisfied, and so they are no longer effective constraints in this case.

**3.** We have been considering Weyl candidates in general, and have shown explicitly that the set of Weyl candidates for which a Lanczos potential can be defined in dimensions  $n$ , where  $n > 4$ , does not exhaust the complete set; since the Weyl curvature tensor is itself a special case from the set of all Weyl candidates we are unable to draw any direct conclusions for *Weyl curvature tensors* from our results above. So the question whether *all Weyl curvature tensors* have a Lanczos potential is still open.

**4.** We have only shown directly that Lanczos potentials for *general* Weyl candidates fail to exist in flat space ( $n \geq 5$ ). In conformally flat space ( $n \geq 5$ ) and constant curvature ( $n \geq 6$ ) we have obtained restrictions on which Weyl candidates may have a Lanczos potential even though we have not proved that these restrictions are effective for all spaces in the respective class. This leaves the possibility open that there are some *special subclasses* of these classes where all Weyl candidates have a Lanczos potential.

**5.** The existence proofs of the Lanczos potential in  $n > 4$  dimensions would need to be established differently than the existence proofs for the four dimensional case since they have to exclude some tensors  $W_{abcd}$  or alternatively exclude some spaces.

Rather than seeing this as an existence problem maybe one should see this as a problem of characterising the class of tensors  $W_{abcd}$  that can be defined through (4) for  $n > 4$  dimensions.

**6.** In a *parallel problem*, regarding the existence of a potential  $\tilde{L}_{abc}$  with the property (2a), for a 4-tensor  $\tilde{W}_{abcd}$  with the properties (3a), (3b) and (3d), Bampi and Caviglia [2] have claimed that such a potential  $\tilde{L}_{abc}$  generally does not exist in spaces of  $n > 6$ , but does exist for spaces of dimension  $n = 4, 5, 6$ . However, it is important to note that although this parallel problem is equivalent to the original Weyl candidate problem when  $n = 4$ , it is *not* equivalent to the original problem when  $n \neq 4$ , and so this parallel result *cannot* be transferred directly to the original problem for a Weyl candidate (or Weyl tensor) when  $n \neq 4$ . Therefore the conclusion — that a Lanczos potential  $L_{abc}$  for the Weyl tensor  $C_{abcd}$  exists only in spaces of dimension 4,5,6 (see Roberts [9], Edgar [3,4]) — which has been drawn from the result of this parallel problem is unjustified.

In fact in the calculations of Bampi and Caviglia [2] leading to the parallel result just mentioned, there seems to be a simple computational error in the very last step of their argument; when this is corrected, their result for the parallel problem should be that a potential does exist generally for spaces of dimension  $n = 4, 5$ , but not for spaces where  $n > 5$ .

The parallel problem considered by Bampi and Caviglia [2] was only a way of getting the results for the original problem. In this paper, as well as in the paper of Bampi and Caviglia, the original problem is the primary target. However, we have deliberately avoided using the extra properties (3c) and (2b) in the original problem compared to the parallel problem; *therefore our results are valid for the parallel problem as well.*

So our results for the parallel problem agree with Bampi and Caviglia for  $n \geq 6$ ; but since we also obtain effective restrictions on  $W_{abcd}$  in the parallel problem for  $n = 5$  we seem to contradict the results of Bampi and Caviglia.

Since Bampi and Caviglia only discuss the cases for  $n > 4$  in passing in their conclusion and also stress that their results in this case are of a generic nature, more work needs to be done to determine precisely their result and its correspondence to ours in 5 dimensions.

**7.** The non-existence of the Lanczos potential, in dimensions  $n > 4$ , should not really come as a surprise. Firstly, the existence proofs of Bampi and Caviglia [2], Illge [6] and Andersson and Edgar [1] depend explicitly and crucially on dimension four; it has proved impossible to write these proofs in a manner which could be used in arbitrary dimension. Secondly, it has been shown by Illge [6], via spinors, and by Edgar [3], via tensors for arbitrary signature in 4 dimensions, that substitution of (1) into the Bianchi equations gives the very simple differential equation

$$\nabla^2 L_{abc} = 0 \tag{26}$$

for zero Ricci tensor and Lanczos gauges. On the other hand, for dimensions  $n > 4$  substitution of (4) into the Bianchi equations gives a much more complicated

differential equation,

$$\nabla^2 L_{abc} + \frac{2(n-4)}{n-2} L_{[a}{}^d{}_{|c;d|b]} - 2L_{[b}{}^{ed} C_{a]dec} + \frac{1}{2} C_{deab} L^{de}{}_c - \frac{4}{n-2} g_{c[a} C_{b]fed} L^{fed} = 0 \quad (27)$$

for zero Ricci tensor and Lanczos gauges (Edgar and Höglund [5]). Equation (26) links up with classical existence and uniqueness results for second order partial differential equations; the more complicated equation (27) is not suitable — because of its second term — for applying these classical results. Thirdly those algorithms in the literature which enable particular Lanczos potentials to be calculated explicitly are constructed in a manner in which the dimension four is crucial (often using spinors); it seems impossible to generalise these algorithms to higher dimensions.

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